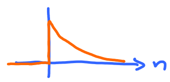
HW2 Q1: 1-D MAP Detector and Exponential Noise

Monday, July 15, 2013 10:58 AM

Exponential Noise

$$f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$



(Note that $\mathbb{E}N = \frac{1}{\lambda}$ and $\text{Var}\,N = \frac{1}{\lambda^2}$)

MATLAB use $\mathbb{E}N$ as the parameter instead of λ

$$P[N > n] = \begin{cases} e^{-\lambda n}, & n \geq 0, \\ \cancel{0}\,1, & \text{otherwise.} \end{cases}$$

In this question, $\lambda = \frac{1}{2}$ and $\mathbb{E}N = \frac{1}{\lambda} = 2$.

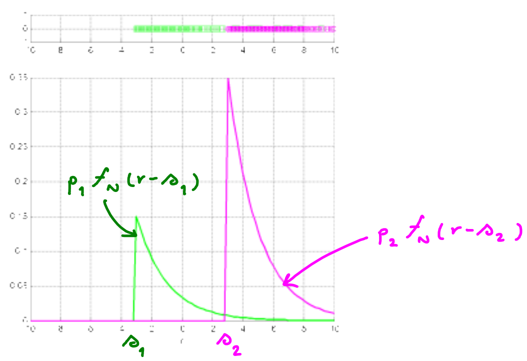
(a)

MAP Detector : Recall that $\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r-s)$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{\uparrow\,0.3}{p_1}\, f_N(r - \underset{\uparrow\,-3}{s_1})$ and $\underset{\uparrow\,0.7}{p_2}\, f_N(r - \underset{\uparrow\,3}{s_2})$



From the graph, it is clear that

1. $R < s_1$ is impossible. So, the detector can do anything in this region without affecting its performance.

2. When $s_1 < r < s_2$, $p_1 f_N(r-s_1) > p_2 f_N(r-s_2)$.
   So, in this region, $\hat{s}_{MAP}(r) = s_1$.

3. When $r > s_2$, $p_1 f_N(r-s_1) < p_2 f_N(r-s_2)$.
   So, in this region, $\hat{s}_{MAP}(r) = s_2$.

Conclusion:

$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & s_1 < r < s_2, \\ s_2, & r > s_2, \\ \text{anything}, & \text{otherwise} \end{cases} \underset{\uparrow\,\text{Simplification}}{=} \begin{cases} s_1, & r < s_2 \\ s_2, & \underset{\uparrow\,\tau^* = s_2}{r \geq s_2} \end{cases}$$

$$= \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

(b) $D_i = \{ r : \hat{s}_{MAP}(r) = s_i \} \Rightarrow \begin{cases} D_1 = (-\infty, 3) \\ D_2 = [3, \infty) \end{cases}$

(c) Probability of error:

In class, we show that for the detector of the form

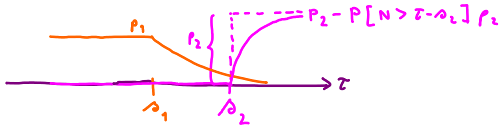
$$\hat{s}(r) = \begin{cases} s_1, & r < \tau, \\ s_2, & r \geq \tau, \end{cases}$$

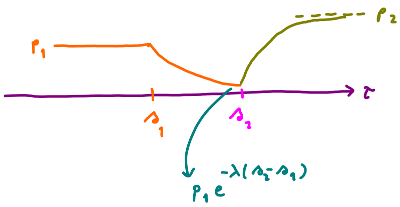
$$P(\mathcal{E}) = p_1 P[N \geq \tau - s_1] + p_2 P[N < \tau - s_2]$$

$$= p_1 P[N \geq \tau - s_1] + p_2 \left(1 - P[N \geq \tau - s_2]\right)$$

For exponential noise, $P[N > n] = \int_n^{\infty} f_N(n)\,dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$

Therefore,





$$P(\mathcal{E}) = \begin{cases} p_1, & \tau < s_1, \\ p_1 e^{-\lambda(\tau - s_1)}, & s_1 \leq \tau \leq s_2, \\ p_1 e^{-\lambda(\tau - s_1)} + p_2\left(1 - e^{-\lambda(\tau - s_2)}\right) & \tau > s_2, \end{cases}$$

$$= \begin{cases} 0.3, & \tau < -3, \\ 0.3\, e^{-\frac{1}{2}(\tau + 3)} & -3 \leq \tau \leq 3, \\ 0.3\, e^{-\frac{1}{2}(\tau+3)} - 0.7\, e^{-\frac{1}{2}(\tau - 3)} + 0.7 & \tau > 3 \end{cases}$$

(d) Method (i), plug-in $\tau = s_2$ into the expression derived in part (c)

$$\Rightarrow P(\mathcal{E}) = p_1 e^{-\lambda(s_2 - s_1)} = 0.3\, e^{-\frac{1}{2}(3-(-3))} = 0.3\, e^{-3} \approx 0.0149$$

Method (ii), $P(\mathcal{E}) = \underbrace{P(\mathcal{E} \mid S = s_1)}_{= P[N > s_2 - s_1]} p_1 + \underbrace{P(\mathcal{E} \mid S = s_2)}_{=0} p_2$

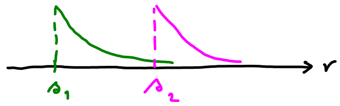
$= P[N > s_2 - s_1]$; $= 0$ because N is always positive and the MAP detector always detect s_2 when $R \geq s_2$

$$= p_1 e^{-\lambda(s_2 - s_1)}$$

(e) Recall that ML detector ignores the prior probabilities; that is

$$\hat{s}_{ML}(r) = \arg\max_{s} f_N(r - s).$$

So, we compare $f_N(r - s_1)$ and $f_N(r - s_2)$



Observe that all the discussion we have in part (a) still works here.

Therefore,

$$\hat{s}_{ML}(r) = \hat{s}_{MAP}(r) = \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

HW2 Q2: 1-D MAP Detector and Uniform Noise

Monday, July 15, 2013 1:33 PM

Recall that the uniform pdf on $[a,b]$ is given by

$$f_N(n) = \begin{cases} \frac{1}{b-a}, & a < n < b, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $a = 4$ and $b = -4$. So,

$$f_N(n) = \begin{cases} 1/8, & -4 < n < 4, \\ 0, & \text{otherwise.} \end{cases}$$

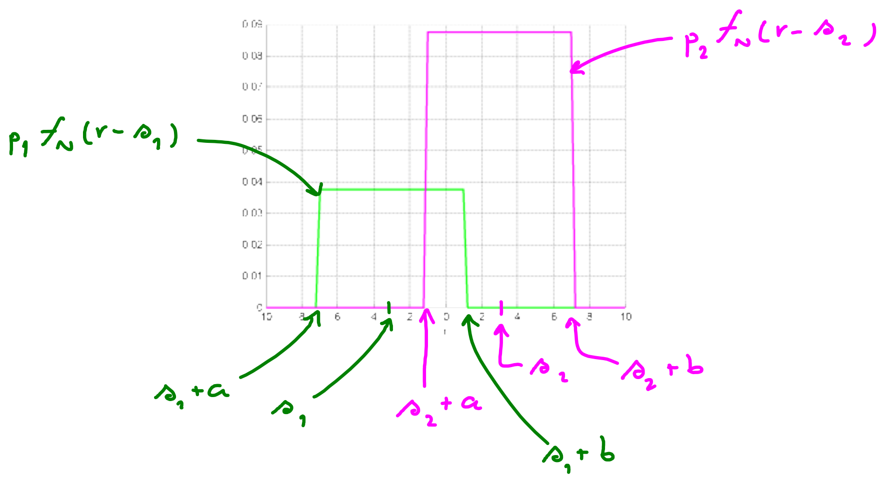
(a) Again, the MAP detector is given by

$$\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r-s).$$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{0.3}{p_1} f_N(r - \underset{-3}{s_1})$ and $\underset{0.7}{p_2} f_N(r - \underset{3}{s_2})$



Observe that

1. when $s_2+a < r < s_2+b$, $p_1 f_N(r-s_1) < p_2 f_N(r-s_2)$.

   Therefore, $\hat{s}_{MAP}(r) = s_2$ in this region.

2. when $s_1+a < r < s_2+a$, $p_1 f_N(r-s_1) > p_2 f_N(r-s_2)$.

   Therefore, $\hat{s}_{MAP}(r) = s_1$ in this region.

3. when $r < s_1+a$ or $r > s_2+b$, the pdf in both cases are 0.

So, these are the impossible regions. The received signal R won't fall in these regions.

Therefore, it does not matter how the detector behaves in this region.

Conclusion: $$\hat{s}_{MAP}(r) = \begin{cases} s_1, & s_1 + a < r < s_2 + a, \\ s_2, & s_2 + a < r < s_2 + b, \\ \text{anything}, & \text{otherwise} \end{cases}$$

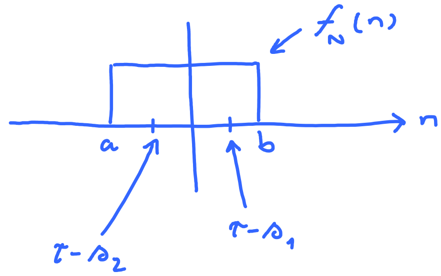
simplification

$$= \begin{cases} s_1, & r < s_2 + a \\ s_2, & r \geq \underbrace{s_2 + a}_{\tau^* = s_2 + a} \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$$

(b) $D_1 = \{ r : \hat{s}(r) = s_1 \} = (-\infty, -1)$

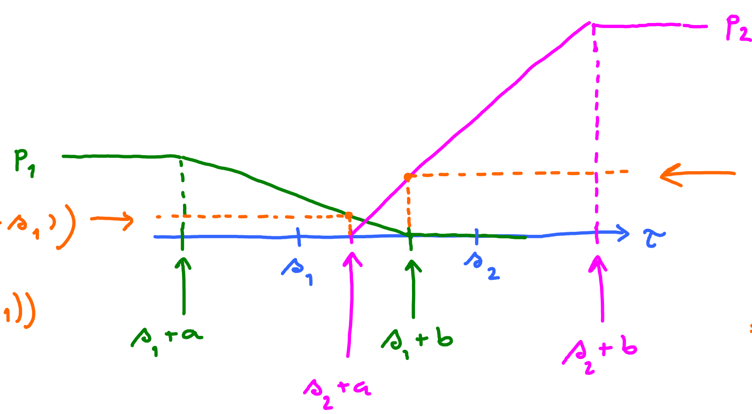
$D_2 = \{ r : \hat{s}(r) = s_2 \} = [1, \infty)$

(c) Recall that $P(\mathcal{E}) = p_1 P[N \geq \tau - s_1] + p_2 P[N < \tau - s_2]$



$$P[N \geq \tau - s_1] = \begin{cases} 1, & \tau - s_1 \leq a \\ \frac{1}{b-a}(b - (\tau - s_1)) & a \leq \tau - s_1 \leq b \\ 0 & \tau - s_1 \geq b \end{cases}$$

$$P[N < \tau - s_2] = \begin{cases} 0, & \tau - s_2 \leq a \\ \frac{1}{b-a}(\tau - s_2 - a) & a \leq \tau - s_2 \leq b \\ 1, & \tau - s_2 \geq b \end{cases}$$



$p_1 \frac{1}{b-a}\left(b - (s_2 + a - s_1)\right) \rightarrow$

$= p_1 \frac{1}{b-a}\left(b - a - (s_2 - s_1)\right)$

$= p_1 \left(1 - \frac{s_2 - s_1}{b-a}\right)$

$\leftarrow p_2 \frac{1}{b-a}\left(s_1 + b - s_2 - a\right)$

$= p_2 \frac{1}{b-a}\left(b - a - (s_2 - s_1)\right)$

$= p_2 \left(1 - \frac{s_2 - s_1}{b-a}\right)$

Here, $p_1 = 0.3$, $p_2 = 0.7$, $s_2 - s_1 = 3 - (-3) = 6$, $b - a = 4 - (-4) = 8$

$s_1 - a = -3 - 4 = -7$ $\quad$ $s_1 + b = -3 + 4 = 1$

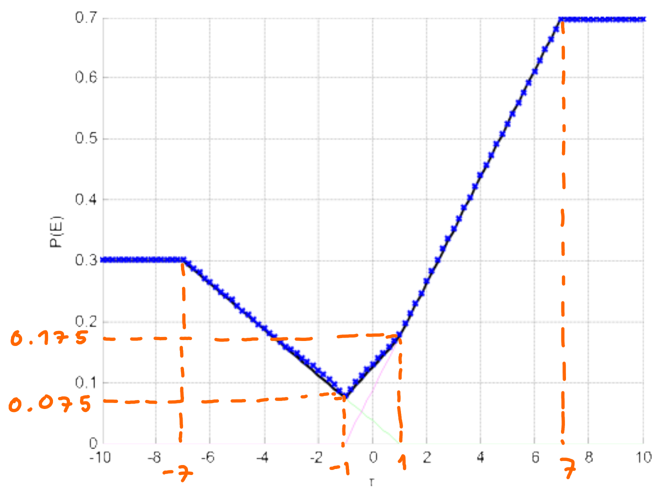
Here, $p_1 = 0.3$, $p_2 = 0.7$, $s_2 - s_1 = 3-(-3) = 6$, $b-a = 4-(-4) = 8$

$$s_1 + a = -3-4 = -7 \qquad s_1 + b = -3+4 = 1$$

$$s_2 + a = 3-4 = -1 \qquad s_2 + b = 3+4 = 7$$

$$p_1\left(1 + \frac{s_2 - s_1}{b-a}\right) = 0.3\left(1-\frac{3}{4}\right) = \frac{0.3}{4} = 0.075$$

$$p_2\left(1 - \frac{s_2 - s_1}{b-a}\right) = 0.7\left(1-\frac{3}{4}\right) = \frac{0.7}{4} = 0.175$$



(d) When $\tau = -1$, pact (c) tells us that $P(\mathcal{E}) = 0.075$.

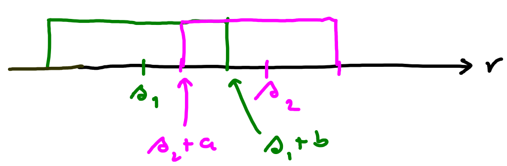
$\uparrow$

τ^* for MAP detector in part (a).

(e) Recall that ML detector ignores the prior probabilities; that is

$$\hat{s}_{ML}(r) = \arg\max_{s} f_N(r-s).$$

So, we compare $f_N(r-s_1)$ and $f_N(r-s_2)$



Observe that when $r < s_2 + a$ or $r > s_1 + b$, we got the same conclusion as in part (a):

$$\hat{s}_{ML}(r) = \begin{cases} s_1, & r < s_2 + a, \\ s_2, & r > s_1 + b. \end{cases}$$

However, when $s_2 + a < r < s_1 + b$, the two graphs are the same.
Therefore, the ML detector is free to choose s_1 or s_2 in this region.

If $p_1 = p_2$, this will not affect the overall $P(\mathcal{E})$. In fact, you will get MAP detector.

Three-point Constellation

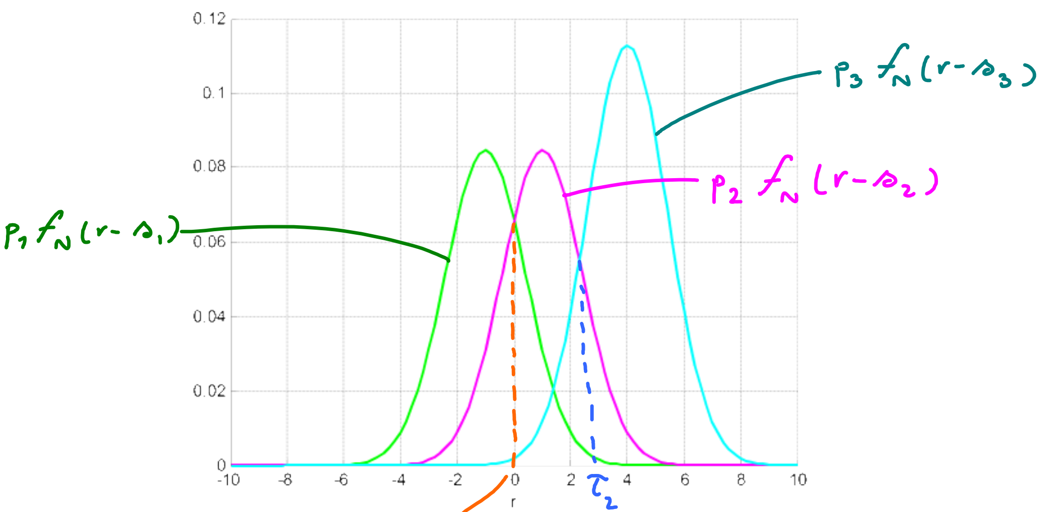
$$S = \{\underset{s_1}{-1}, \underset{s_2}{1}, \underset{s_3}{4}\}$$

$$\underbrace{P[S=-1]}_{p_1} = 0.3 = \underbrace{P[S=1]}_{p_2}$$

$$\underbrace{P[S=4]}_{p_3} = 0.4$$

Average energy per symbol $= (-1)^2 \times 0.3 + 1^2 \times 0.3 + 4^2 \times 0.4$

$= 0.3 + 0.3 + 6.4 = 7$



To find τ_1, we find r such that $p_1 f_N(r-s_1) = p_2 f_N(r-s_2)$.

$$p_1 \cancel{\frac{1}{\sqrt{2\pi}\Delta}} e^{-\frac{1}{2}\left(\frac{r-s_1}{\Delta}\right)^2} = p_2 \cancel{\frac{1}{\sqrt{2\pi}\Delta}} e^{-\frac{1}{2}\left(\frac{r-s_2}{\Delta}\right)^2}$$

$$\frac{p_1}{p_2} = e^{-\frac{1}{2\Delta^2}\left((r-s_2)^2 - (r-s_1)^2\right)}$$

$$2\Delta^2 \ln \frac{p_1}{p_2} = (2r - s_2 - s_1)(s_2 - s_1)$$

$$r = \frac{\Delta^2}{s_2 - s_1} \ln \frac{p_1}{p_2} + \frac{s_1 + s_2}{2}$$

$\underbrace{\left(s_2 - s_1 \qquad p_2 \qquad 2 \right.}$

This is the same formula that we derived in lecture to find τ^* for the MAP detector when $M=2$.

Here, $p_1 = p_2$. So, $\tau_1 = \frac{s_1 + s_2}{2} = \frac{-1+1}{2} = 0$

Similarly, we can find τ_2 by $\frac{\Delta^2}{s_3 - s_2} \ln \frac{p_3}{p_2} + \frac{s_2 + s_3}{2} = 2.3082$

The MAP detector is given by

$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & r \le \tau_1 \\ s_2, & \tau_1 < r \le \tau_2 \\ s_3, & r > \tau_2 \end{cases}$$

$$= \begin{cases} -1, & r \le 0 \\ 1 & 0 < r \le 2.3082 \\ 4 & r > 2.3082 \end{cases}$$

$$D_1 = \{ r : \hat{s}_{MAP}(r) = s_1 \} = (-\infty, 0]$$

$$D_2 = \{ r : \hat{s}_{MAP}(r) = s_2 \} = (0, 2.3082]$$

$$D_3 = \{ r : \hat{s}_{MAP}(r) = s_3 \} = (2.3082, \infty)$$

$$P(\mathcal{E}) = \underbrace{P(\mathcal{E} \mid S = s_1)}_{= P[R > \tau_1 \mid S = s_1]} p_1 + \underbrace{P(\mathcal{E} \mid S = s_2)}_{P[R \le \tau_1 \mid S = s_2] + P[R > \tau_2 \mid S = s_2]} p_2 + \underbrace{P(\mathcal{E} \mid S = s_3)}_{P[R \le \tau_2 \mid S = s_3]} p_3$$

$$= P[N > \tau_1 - s_1]\, p_1 + \left(P[N \le \tau_1 - s_2] + P[N > \tau_2 - s_2] \right) p_2 + P[N \le \tau_2 - s_3]\, p_3$$

$$= p_1 Q\left(\frac{\tau_1 - s_1}{\Delta} \right) + \left(Q\left(\frac{s_2 - \tau_1}{\Delta} \right) + Q\left(\frac{\tau_2 - s_2}{\Delta} \right) \right) p_2 + p_3 Q\left(\frac{s_3 - \tau_2}{\Delta} \right)$$

$$= 0.2434$$

HW2 Q4: 1-D Multi-Level MAP Detector and Gaussian Noise

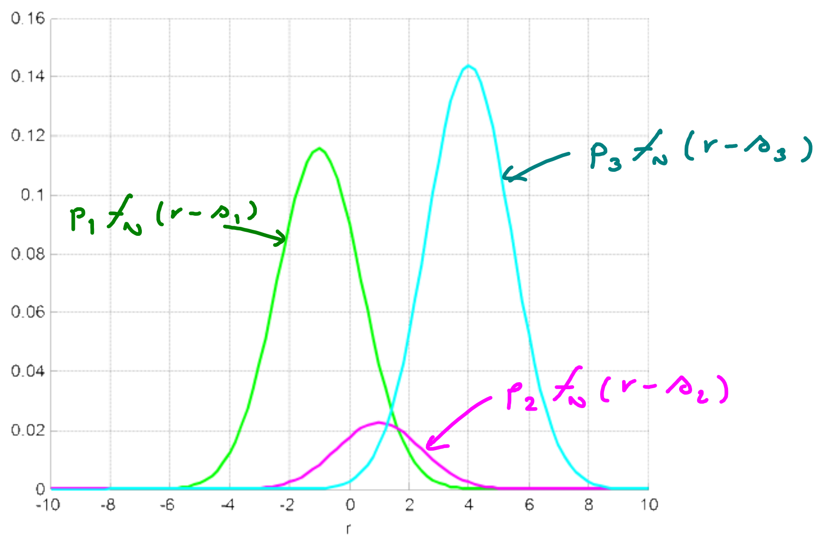
Wednesday, July 24, 2013 3:13 PM

(a) Average symbol energy:

$$E_s = 0.41 \times (-1)^2 + 0.08 \times (1)^2 + 0.51 \times 4^2 = 8.65$$

(b)

Note that $p_2 f_N(r - s_2)$ is always below $p_1 f_N(r-s_1)$ and $p_3 f_N(r-s_3)$.



So, the MAP detector will never choose s_2 as its answer.

Therefore, $P(\mathcal{E} \mid S = s_2) = 1$.

Note that although this may seem bad, it is in fact not so bad. The key idea is to realize that p_2 is very small and hence the contribution to the overall $P(\mathcal{E})$ is also small.

HW2 Q5: 1-D Standard Multi-Level MAP Detector and Expo Noise

Wednesday, July 24, 2013 3:35 PM

(a) Here there are four ($M=4$) possible values for s : $-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}$

They are equally likely. Therefore $p_i = \frac{1}{4}$.

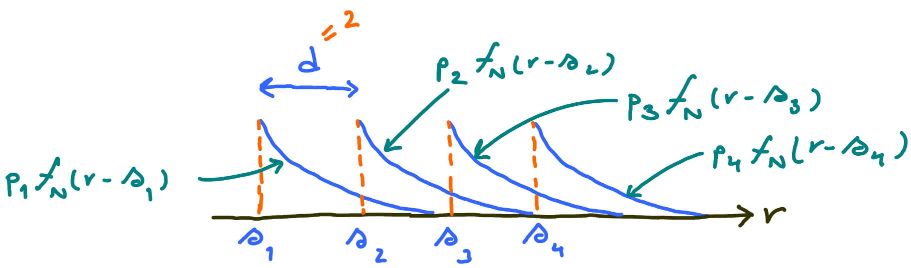
$$E_s = \sum_i p_i |s_i|^2 = \frac{1}{4}\left(\left(-\frac{3}{2}d\right)^2 + \left(-\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{3}{2}d\right)^2\right)$$

$$= \frac{1}{16} d^2 (9+1+1+9) = \frac{20}{16} d^2 = \frac{5}{4} d^2$$

(b) Each symbol communicates $\log_2 M = \log_2 4 = 2$ bits.

Therefore, energy per bit $E_b = \frac{E_s}{2} = \frac{5}{8} d^2$

(c)



$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & r < s_2 \\ s_2, & s_2 \le r < s_3 \\ s_3, & s_3 \le r < s_4 \\ s_4, & s_4 \le r \end{cases} = \begin{cases} -\frac{3d}{2}, & r < -\frac{d}{2} \\ -\frac{d}{2}, & -\frac{d}{2} \le r < \frac{d}{2} \\ \frac{d}{2}, & \frac{d}{2} \le r < \frac{3d}{2} \\ \frac{3d}{2}, & \frac{3d}{2} \le r \end{cases}$$

(d)

$$P(\mathcal{E} \mid S = s_i) = P[N > d] = e^{-\lambda d} \quad \text{for } i = 1, 2, 3$$

$$P(\mathcal{E} \mid S = s_4) = 0$$

$$P(\mathcal{E}) = \frac{3}{4} e^{-\lambda d}.$$

(e)

For comparable noise "power" with the Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$

we choose $\frac{1}{\lambda^2} = \sigma^2 \Rightarrow \frac{1}{\lambda} = \sigma$

mean of the exponential noise

mean of the exponential noise

From part (b), $E_b = \frac{5}{8} d^2$. So, $d = \sqrt{\frac{8}{5} E_b}$.

From part (d), $P(\mathcal{E}) = \frac{3}{4} e^{-\lambda d} = \frac{3}{4} \exp\left(-\frac{1}{\Delta}\sqrt{\frac{8}{5} E_b}\right) = \frac{3}{4} \exp\left(-2\sqrt{\frac{2}{5}\frac{E_b}{\Delta^2}}\right)$

This is the plot when the noise is Gaussian (which we derived in class).